ON ABELIAN SUBGROUPS OF FINITELY GENERATED METABELIAN GROUPS

V. H. MIKAELIAN   A. YU. OLSHANSKII

Abstract. In this note we introduce the class of $\mathcal{H}$-groups (or Hall groups) related to the class of $\mathcal{B}$-groups defined by Ph. Hall in 1950's. Establishing some basic properties of Hall groups we use them to obtain results concerning embeddings of abelian groups. In particular, we give an explicit classification of all abelian groups that can occur as subgroups in finitely-generated metabelian groups. Hall groups allow to give a negative answer to the Baumslag's conjecture of 1990 on the cardinality of the set of isomorphism classes for abelian subgroups in finitely generated metabelian groups.

1. Introduction

Our note goes back to the old paper of P. Hall [7] who obtained the properties of abelian *normal* subgroups in finitely generated metabelian and abelian-by-polycyclic groups. Let $\mathcal{B}$ be the class of all abelian groups B, where B is an abelian normal subgroup of some finitely generated groups G with polycyclic quotient G/B. It is proved in Lemmas 8 and 5.2 of [7], that $\mathcal{B} \subset \mathcal{H}$, where the class of countable abelian groups $\mathcal{H}$ can be defined as follows (in the present paper, we will say that the groups from $\mathcal{H}$ are *Hall groups*). By definition, $H \in \mathcal{H}$ if

(1) H is a (finite or) countable abelian group,

(2) $H = T \oplus K$, where T is a bounded torsion group (i.e., the orders of all elements in T are bounded), K is torsion free, and

(3) K has a free abelian subgroup F such that K/F is a torsion group with trivial p-subgroups for all primes except for the members of a finite set $\pi = \pi(K)$.

Applying Ph. Hall's results we observe the following.

Theorem 1. *A group H is an abelian subgroup of a finitely generated abelian-by-polycyclic group if and only if H is a Hall group. Moreover, every Hall groups is embeddable into the derived subgroup of a 2-generated metabelian group.*

Since by Baumslag - Remeslennikov theorem [1, 12] every finitely generated metabelian group is embeddable into finitely presented metabelian group, we have

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Corollary 1. *Let H be an abelian group. The following properties are equivalent.*

(1) H is a subgroup of a finitely generated metabelian group;

(2) H is a subgroup of finitely generated abelian-by-polycyclic group;

(3) H is a subgroup of finitely presented metabelian group;

(4) H is a subgroup of a 2-generated metabelian group;

(5) H is a Hall group.

Recall that G. Baumslag, U. Stammbach, and R. Strebel [2] proved that the 2-generated free metabelian group contains uncountably many non-isomorphic subgroups. On the other hand, in 1990, reflecting on Hall's results of [7], Gilbert Baumslag [3] wrote: *"It is easy to see that the abelian subgroups of a free metabelian group are free abelian. So there are only a countable number of isomorphism classes of abelian subgroups of a free metabelian group of finite rank. It appears likely that this observation holds also for finitely generated metabelian groups as a whole, but I have not yet managed to prove this."* Unfortunately this conjecture of Baumslag is unprovable because using Theorem 1 we obtain:

Theorem 2. *There is a 2-generated metabelian group containing continuously many pairwise non-isomorphic abelian subgroups.*

It follows from Theorem 2 that the class of Hall groups $\mathcal{H}$ is bigger than the class $\mathcal{B}$ since the set of non-isomorphic groups of $\mathcal{B}$ is countable. And we in fact are able to explicitly build and example of a Hall group which is not a $\mathcal{B}$-group (see Section 5).

V.A. Roman'kov [14, 15] proved that every finitely generated nilpotent (every polycyclic) group is a subgroup of a 2-generated nilipotent (resp. polycyclic) group. In Sections 6, 7 we present two "thrifty" embeddings of finitely generated abelian groups into 2-generated metabelian, polycyclic groups, and also examples of countable groups of solvable length $l > 1$ non-embeddable in finitely generated (l+1)-solvable groups, and two open questions.

Remark 1. Let K be the subgroup from the above definition of Hall group. Since it is torsion free and countable, K is embeddable into a countable direct power $\bigoplus \mathbb{Q}_i$ of the additive group of the rationals. Moreover, the basis elements of the subgroup F can be mapped to the vectors $(0, \dots, 0, 1, 0, \dots)$ with one non-zero coordinate. Therefore K becomes a subgroup of the group $\bigoplus_i (D_n)_i$, where D_n consists of all rationals of the form m/n^k, with n equal to the product of all primes from the set π. This observation and the property that the class $\mathcal{H}$ is closed under subgroups (Lemma 4.2 [7]) imply the equality $\mathcal{H} = \cup_{n=1}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n$ is the class of subgroups in the countable direct power of the group $\mathbb{Z}_n \oplus D_n$. (We use here that every bounded abelian group is embeddable in a direct sum of copies of $\mathbb{Z}_n$ for some n [13].)

2. Necessity

Lemma 1. *Let H be an abelian subgroup of a finitely generated abelian-by-polycyclic group G. Then H is a Hall group.*

Proof. Let B be an abelian normal subgroup of G with polycyclic quotient $P = G/B$. By the inclusions $B \in \mathcal{B} \subset \mathcal{H}$, we have that the torsion subgroup S of B and $H \cap S$ are bounded groups. The quotient $H/(H \cap B) \cong HB/B$ is finitely generated being a subgroup of the polycyclic group P, and so the torsion subgroup of it also is bounded. It follows that the torsion subgroup T of H is bounded as well, and so $H = T \oplus K$ by Kulikov's theorem [5, 13]. The torsion free group K can be now regarded as a subgroup of the group G/S with torsion free $C = B/S$.

Since P and its abelian subgroup $K/(K \cap C)$ are polycyclic, we have a series of subgroups $K_1 \leq K_2 \leq K$ with torsion free $K_1 = K \cap C$, free abelian K_2/K_1, and finite K/K_2. Since $C \in \mathcal{B} \subset \mathcal{H}$, we have that $K_1 \in \mathcal{H}$ by Remark 1, and moreover $K_1 \in \mathcal{H}_n$ for some n. Since K_2/K_1 is free abelian, we have an isomorphism $K_2 \cong K_1 \oplus (K_2/K_1)$, and therefore $K_2 \in \mathcal{H}_n$ holds. In turn, this implies that K itself belongs to some $\mathcal{H}_m$ because it is torsion free, and the free subgroup F of K_2 (from the definition of Hall group applied to K_2) will work for K as well (but with a bigger finite set π') since $K/K_2 \leq \infty$.

Thus $H = T \oplus K$ with the required properties of T and K. □

Remark 2. It is seen from the proof, that $H \in \mathcal{H}_k$, where k depends only on G.

3. The embedding of $(D_n)^\infty$

Let $D(n)$ be the countable direct power of the group D_n from Remark 1.

Lemma 2. *The group $D(n)$ embeds into the derived subgroup of a 2-generated metabelian group G. In addition, $G/[G,G]$ is a free abelian group of rank 2.*

Proof. Let us regard the elements of $D(n)$ as vectors $\mathbf{x} = (\dots, x_{-1}, x_0, x_1, \dots)$ with finitely many nonzero coordinates $x_i = m_i/n^{k_i}$ $(i, m_i, k_i \in \mathbb{Z}, k_i \geq 0)$. We will use two automorphisms α and β of the additive group $D(n)$, namely, α shifts the coordinates, i.e., maps arbitrary vector $\mathbf{x}$ to $\mathbf{y}$, where $y_i = x_{i-1}$ $(i \in \mathbb{Z})$, and β multiplies all the vectors by n: $\beta(\mathbf{x}) = n\mathbf{x}$.

Since the automorphisms α and β commute, we can build a split extension S of $D(n)$ by a free abelian group $A = \langle a, b \rangle$ acting by conjugation as follows: $a^{-1}\mathbf{x}a = \alpha(\mathbf{x}), b^{-1}\mathbf{x}b = \beta(\mathbf{x})$. Since $b\mathbf{x}b^{-1} = \mathbf{x}/n$ for any $\mathbf{x} \in D(n)$, we see that the normal closure in S of the vector $\mathbf{c} = (\dots, 0, 1, 0, \dots)$, where $1 = c_0$, is the entire subgroup $D(n)$, and so $S = \langle a, b, \mathbf{c} \rangle$.

Now compute the commutator $\mathbf{d} = [b\mathbf{c}, a]$ using the additive notation for the elements from $D(n)$:

$$\mathbf{d} = (-\mathbf{c})b^{-1}a^{-1}b\mathbf{c}a = (-\mathbf{c})a^{-1}\mathbf{c}a = (-\mathbf{c}) + (a^{-1}\mathbf{c}a),$$

and so $d_0 = -1, d_1 = 1$, and other coordinates of $\mathbf{d}$ are 0. Denote by G the metabelian subgroup generated by a and $f = b\mathbf{c}$. It contains $\mathbf{d}$ in the derived subgroup $[G,G]$ and, for positive integers k-s, contains all the elements $(f)^k\mathbf{d}(f)^{-k} = b^k\mathbf{d}b^{-k} = \mathbf{d}/n^k$. Therefore $[G,G]$ contains a copy D' of the group D_n, and all coordinates of the vectors $\mathbf{z}$ from D' are zero except for z_0 and z_1. It follows that the shifts $a^{-l}D'a^l$ with even exponents l-s generate in $[G,G]$, a subgroup isomorphic with $D(n)$.

To prove the last statement of the lemma, we observe that the projection of the split extension S onto the free abelian group A maps the generators a and f of the group G to the free generators of A. It induces an epimorphism of the 2-generated group $G/[G,G]$ onto A that obviously must be an isomorphism. □

4. Embeddings of arbitrary Hall groups

We denote by $Z(n)$ the countable direct power of a cyclic group of order n. Let $\mathfrak{A}_m$ be the variety of abelian groups of exponents dividing the positive integer m and let $A_m(X)$ be the verbal subgroup of a group X corresponding to the variety $\mathfrak{A}_m$, i.e., $A_m(X) = [X,X]X^m$ [11].

Lemma 3. *Let F be a free group and L a normal subgroup of infinite index in F. Assume that the quotient $L/A_m(L)$ has an element g of order $n \geq 1$. Then the normal closure of this element in $F/A_m(L)$ has a subgroup isomorphic to $Z(n)$.*

Proof. Since the variety $\mathfrak{A}_m$ is abelian, the verbal $\mathfrak{A}_m$-products of groups (see [11, 16]) coincide with standard direct (restricted) wreath products. Therefore by Magnus - Shmelkin's theorem [16], the groups $Y = F/A_m(L)$ embeds in the wreath product $V = Bwr(F/L)$, where $B \cong F/A_m(F)$. Recall that V is the semidirect product CW, where $C = F/L$ and W is the direct power of B with the right regular action of the group C by conjugation on the set of direct factors. (The direct factors $B(c)$ of W are isomorphic to B an indexed by the elements $c \in C$.) The embedding enjoys the properties: $YW = V$ and (the image of) $L/A_m(L)$ becomes a subgroup of W.

Thus, (under the embedding) the element g belongs to a product $B(c_1) \times \cdots \times B(c_t)$ for a finite set $\{c_1, \ldots, c_t\} \subset C$. Since C is infinite, there is $d \in C$ such that the support $\{c_1 d, \ldots, c_t d\}$ of the conjugate elements $g_1 = d^{-1} g d$ is disjoint with the set $\{c_1, \ldots, c_t\}$. Recall that $d = yw$ for some $y \in Y$, $w \in V$. Since W is abelian, we have $g_1 = ygy^{-1}$, and so g_1 belongs to the normal closure of g in Y, and it generates, together with g, the direct product of two cyclic groups of order n. Keep choosing elements $g_2, g_3, \ldots$ in this manner, one can obtain the elements of the normal closure of g in Y, which generate a subgroup isomorphic to $Z(n)$. □

Lemma 4. *The abelian group $Z(n) \oplus D(n)$ embeds into the derived subgroup of a 2-generator metabelian group M.*

Proof. We may assume that $n > 1$ (e.g., by Lemma 2).

Let F_2 be a 2-generated free group. The 2-generated group G from Lemma 2 can be presented as $G \cong F_2/N$, where the normal subgroup N contains the second derived subgroup F_2'' since G is metabelian. On the other one hand, $N \neq F_2''$ since the quotient F_2/F_2'' is an extension of the free abelian group F_2'/F_2'' by the free abelian group F_2/F_2', and so it cannot contain an element divisible by all powers n^k for $n \geq 2$. Observe also that $N \leq F_2'$ since F_2/N maps onto F_2/F_2' by the second statement of Lemma 2. Hence N/F_2'' is a non-trivial subgroup of the free abelian group F_2'/F_2''.

Since the group F_2'/F_2'' is free abelian, the intersection $\cap_{m \in I} A_m(F_2'/F_2'')$ is trivial for arbitrary infinite set I of positive integers, i.e., $\cap_{m \in I} A_m(F_2') = F_2''$. Denote

$N_m = N \cap A_m(F')$. Since

$$N/N_m \cong NA_m(F') \,/\, A_m(F') \leq F'/A_m(F') \in \mathfrak{A}_m,$$

the group N/N_m has finite exponent dividing m. Taking large enough powers of a prime p as the values of m we can get elements with infinitely growing orders in the groups N/N_m, for, if the orders of all elements of N/N_m for all $m = p^k$ were restricted from above by a single number, then the quotient $N/(\cap_{k=1}^{\infty} N_{p^k})$ would have a finite exponent, a contradiction with the facts that $\cap_{k=1}^{\infty} N_{p^k} \leq \cap_{k=1}^{\infty} A_{p^k}(F_2') = F_2''$ and N/F'' is a nontrivial torsion free group. Hence for a given $k \geq 1$, we can take m large enough so that the exponent of the group N/N_m to be divisible by p^k. Since one can choose such $m_1, ..., m_s$ for every prime-power divisor p^{k_i} of n, there is $m = m_1 \ldots m_s$ such that the abelian group N/N_m has an element of order n.

The group N/N_m is isomorphic to the normal subgroup $NA_m(F_2') \,/\, A_m(F_2')$ of $F_2/A_m(F_2')$. By Lemma 3 (with $L = F_2'$), both these groups contain subgroups isomorphic with $Z(n)$.

By the choice of G, the abelian derived subgroup of the 2-generator metabelian group $M = F_2/N_m$ contains a subgroup $U = R/N_m$ such that $R/N \cong D(n)$. Therefore the subgroup $T = N/N_m$ contains all torsion elements of U, has exponent $\leq m$, and by Kulikov's theorem [13, 5], $U \cong T \times D(n)$. Since T contains a subgroup $Z(n)$, the group M' contains a subgroup isomorphic with $Z(n) \times D(n)$, as required. □

Proof of Theorem 1 The statements of the theorem follow from Lemmas 1, 4, and Remark 1. □

5. Non-isomorphic subgroups of $D(p)$

The set of non-isomorphic groups B in the class $\mathcal{B}$ is countable. Indeed Ph. Hall observed [7] that each B is a finitely generated module over the group ring $\mathbb{Z}P$ of a polycyclic group P and he proved that such modules are Noetherian. Thus the set of non-isomorphic $\mathbb{Z}P$-modules is countable, and it suffices to take into account that the set of non-isomorphic polycyclic groups also is countable.

We want to prove that for every prime p, the group $D(p)$ has $2^{\aleph_0}$ non-isomorphic subgroups, which together with Theorem 1 prove Theorem 2. Our proof is based on Chapter XIII of Fuchs' book [5].

The system of torsion free abelian groups $\{G_i | i \in I\}$ is said to be *rigid* if $Hom(G_i, G_i) \leq \mathbb{Q}$, and $Hom(G_i, G_j) \cong 0$ for any $i, j \in I$, $i \neq j$. In other words, each endomorphism of a group in a rigid system is a multiplication by a rational number, and there only is zero homomorphism between two distinct groups of the system. An example of a rigid system of continuum cardinality is constructed in Example 5, Section 88 [5]. Given a prime p and $r \geq 2$, take $r-1$ algebraically independent (over rational field $\mathbb{Q}$) p-adic units $\pi_2, \ldots, \pi_r$, and take an extra $\pi_1 = 1$. Let π_{in} be the $n-1$-th partial sum of the canonic presentation of π_i:

$$\pi_i = s_{i0} + s_{i1}p + \cdots + s_{in}p^n + \cdots \quad (0 \leq s_{in} < p).$$

Denote:

$$x_n = p^{-n}(a_1 + \pi_{2n}a_2 + \cdots + \pi_{rn}a_r),$$

where $a_1, \ldots, a_r$ is a basis of in the vector space $\mathbb{Q}^r$, and take the group:

$$A_{\pi_2,\ldots,\pi_r} = \langle a_1, \ldots, a_r,\, x_1, \ldots, x_n, \ldots \rangle.$$

From the construction of x_n above it is clear that $A_{\pi_2,\ldots,\pi_r}$ is a subgroup in the power $D_p^r \le D(p)$. As it easily follows from the properties of p-adic numbers, taking a bigger set of algebraically independent p-adic units $\pi_2, \ldots, \pi_s, \pi'_2, \ldots, \pi'_r$ we get that $Hom(A_{\pi_2,\ldots,\pi_r}, A_{\pi'_2,\ldots,\pi'_r}) = 0$ (see Example 5, Section 88 [5]). In particular, since there are continuously many algebraically independent p-adic units, we have $2^{\aleph_0}$ pairwise non-isomorphic subgroups in D_p^r.

Thus Theorem 2 is proved: $2^{\aleph_0}$ pairwise non-isomorphic subgroups $A_{\pi_2,\ldots,\pi_r}$ are contained in in $D(p)$, and the latter has an isomorphic embedding in a 2-generator metabelian group by Lemma 4.

Theorem 2 also means that the classes of $\mathcal{B}$-groups and of Hall groups are different, although there is similarity in their definitions (a $\mathcal{B}$-group is the Hall group, which is not only embeddable into a finitely generated abelian-by-polycyclic group, but also has a normal embedding in such a group). In particular, it is easy to see that these two classes coincide over torsion groups. Let us conclude this section by an explicit example of a Hall group which is not a $\mathcal{B}$-group.

Example 1. Let G be the group built in 4.4.2 in [13]. G is an indecomposable abelian group of rank two, and is defined as a subgroup in 2-dimensional vector space $\mathbb{Q}^2$ with basis u, v as follows: for three distinct primes p, q, r set G to be the subgroup of $\mathbb{Q}^2$ generated by all elements:

$$p^m u, \quad q^m, \quad r^m(u+v)$$

for all integer values m. Evidently G is a Hall group, since it is subgroup in D_{pqr}^2.

Assume that $G \in \mathcal{B}$, i.e., we are able to normally embed G into an abelian-by polycyclic finitely generated group M. Then the centralizer $C = C_M(G)$ contains the abelian group G and has index $[M : C] \le 2$ since M/C faithfully acts on G by conjugation automorphisms, and the only automorphisms of G are $\pm id_G$ (see 4.4.2 in [13]). Therefore C is also a finitely generated abelian-by polycyclic group, and by [7], G is a finitely generated C-module. Since the action of C on G is trivial, G has to be a finitely generated abelian group, a contradiction.

6. Embeddings of finitely generated abelian groups

Proposition 1. *Every finitely generated abelian group H is isomorphic to the center of a 2-generated metabelian, nilpotent group G. If H is finite then so is G.*

Proof. Let F be a free group of free rank 2 in the variety of metabelian and nilpotent of class $\le c$ groups ($c \ge 2$). The c-th member C of the lower central series of F is contained in the center of F. Furthermore, by Corollary 36.23 of [10], C is a free abelian group of rank $c-1$.

Given an n-generated abelian group H, it can be presented as a quotient C/B if $c - 1 \ge n$, and so the group H is isomorphic to a central factor L/K of F. Since a finitely generated nilpotent group satisfies the ascending chain condition for subgroups [13], we can assume that L is a maximal normal subgroup of F such

that $H \cong L/K$ for some normal subgroups K of F and L/K is a central subgroup in F/K.

Now we want to prove that L/K is the center of $G = F/K$. Arguing by contradiction, we have a bigger central subgroup $A = M/K$ in F/K, i.e. $M > L$. But every subgroup of a finitely generated abelian group A is isomorphic to a factor group of A (trivially follows from [4, Theorem 15.6]). Hence the subgroup $H \cong L/K \leq A$ is isomorphic to a central factor M/N of F, contrary to the maximality of L.

If H is finite then so is G since the center of an infinite finitely generated, nilpotent group is infinite [8, Exercise 17.2.10]. The theorem is proved. □

Proposition 2. *Every finitely generated abelian group H is a normal subgroup with a finite cyclic quotient G/H in a 2-generated metabelian group G.*

Proof. The group H is the direct sum of cyclic subgroup

$$\langle z_1 \rangle \oplus \cdots \oplus \langle z_m \rangle \oplus \langle z_{m+1} \rangle \oplus \cdots \oplus \langle z_{m+n} \rangle \quad (m, n \geq 0),$$

where $z_1, \ldots, z_m$ have infinite orders, and the order n_{i+1} of z_{i+1} divides the finite order n_i of z_i for $i = m+1, \ldots, m+n-1$.

Then mapping φ on the generators of H given by the rules

$$\begin{aligned} &z_1 \mapsto z_2,\ z_2 \mapsto z_3, \ldots, z_{m-1} \mapsto z_m, \\ &z_m \mapsto z_1 z_{m+1}, z_{m+1} \mapsto z_{m+1} z_{m+2}, \ldots, z_{m+n-1} \mapsto z_{m+n-1} z_{m+n}, \\ &z_{m+n} \mapsto z_{m+n} \end{aligned} \tag{1}$$

extends to the endomorphism of H since it is well defined on the direct summand due to the conditions $n_{i+1} | n_i$. This endomorphism (denoted also by φ) is surjective since it is surjective on the φ-invariant subgroup $L = \langle z_{m+1}, \ldots, z_{m+n} \rangle$ and on the quotient H/L. By the Hopf property of H, φ is its automorphism.

Furthermore, φ^m induces the identity automorphism of H/L, and φ^{ms} is identical on L for some $s > 0$ since the subgroup L is finite. It follows that $\varphi^{msl} = id_H$, where l is the order of L.

Let now G be the semidirect product of H and the finite cyclic group $\langle \varphi \rangle$ with the defined above action of φ. Applying the conjugations by the powers of φ to z_1, we successively obtain from (1) that $z_2, \ldots, z_m, z_{m+1}, \ldots, z_{m+n} \in \langle \varphi, z_1 \rangle$, that is, G is a 2-generated group, as desired. □

7. Further discussion

The fact that countable abelian, nilpotent, or generalized nilpotent groups are not in general embeddable into finitely generated groups from the mentioned classes, is a background of the embedding theorems for countable solvable groups: solvability, in some sense, is the "first property" that can be added to embeddings of countable groups into finitely-generated groups.

One of the main motives in theory of embeddings of groups is: Given an embedding of the group H into the group G, then "how close" is the group G to H? By the well-known result of B.H. Neumann and Hanna Neumann [10], every countable solvable group of solvability length l is embeddable into a 2-generated

solvable group of length $l+2$ but not, in general, of $l+1$. To show that $l+2$ may not be replaced by $l+1$, Ph. Hall [6] (Lemma 2) and B.H. Neumann and Hanna Neumann [10] (Lemma 5.3) bring explicit examples for $l=1$, namely the additive group of rational numbers $\mathbb{Q}$ and the quasicyclic groups $C(p^\infty)$, resp. That lemmas can be generalized in one more direction: counter-examples of the above mentioned type can be found for any $l \geq 1$.

Example 2. Denote by H the group of all upper unitriangular $n \times n$ matrices over $\mathbb{Q}$. It is easy to check that H is a (uniquely) divisible group, and it is solvable of prescribed length $l \geq 1$ for $n = 2^{l-1}+1$. The group H does not embed into a finitely generated group G of solvable length $l+1$. Indeed, the finitely generated abelian group G/G' has no non-trivial divisible subgroups, i.e., HG'/G' is trivial, and so $H \leq G'$. Note that no non-trivial subgroup of G'/G'' is divisible because G'/G'' is a Hall group by [6]. Hence HG''/G'' is trivial, i.e., $H \leq G''$, and we come to a contradiction since the length of G'' is at most $l-1$.

Problem 1. *For any $l \geq 2$ obtain an embedding criterion (similar to that given in Theorem 1 for $l=1$): Which countable groups of solvable length l are embeddable in finitely generated solvable groups of length $l+1$?*

If in case $l=1$, one replaces metabelian groups by a slightly larger class of center-by-metabelian groups, then all the restrictions on H will be removed, since Ph. Hall [6, Theorem 6] showed that every countable abelian group H is the center of a 2-generated center-by-metabelian group. On the other hand, the examples of Section 6 demonstrate that finitely generated abelian groups admit embeddings in 2-generated metabelian groups with nice additional properties. What can we expect if $l \geq 2$? The sharper formulation:

Problem 2. *Does every finitely generated solvable group of length $l \geq 2$ embed into a 2-generated solvable group of length $l+1$? Or at least, into some k-generated $(l+1)$-solvable group, where $k = k(l)$?*

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Vahagn H. Mikaelian:
Department of Applied Mathematics
Yerevan State University
Yerevan 0025, Armenia.
E-mail: v.mikaelian@gmail.com

Alexander Yu. Olshanskii:
Department of Mathematics
Vanderbilt University
Nashville, TN 37240, USA.
E-mail: alexander.olshanskiy@vanderbilt.edu